

Solutions to Coursework #7

Q1

(a) Using σ_u, σ_v from Q3 ^{of} ~~HW~~ CWB:

$$\begin{aligned} \textcircled{5} E = \sigma_u \cdot \sigma_u &= \|\sigma_u\|^2 = b^2 \sin^2 u \cos^2 v + b^2 \sin^2 u \sin^2 v \\ &\quad + b^2 \cos^2 u \\ &= b^2 \sin^2 u + b^2 \cos^2 u = b^2 \end{aligned}$$

$$\begin{aligned} \textcircled{5} G = \sigma_v \cdot \sigma_v &= \|\sigma_v\|^2 = (a + b \cos u)^2 (\sin^2 v + \cos^2 v) \\ &= (a + b \cos u)^2 \end{aligned}$$

$$\begin{aligned} \textcircled{5} F = \sigma_u \cdot \sigma_v &= b(a + b \cos u) (-\sin u \cos v (-\sin v) \\ &\quad - \sin u \sin v \cos v) \\ &= 0 \end{aligned}$$

$$\text{so } ds^2 = b^2 du^2 + (a + b \cos u)^2 dv^2$$

$$\begin{aligned} \textcircled{5} (b) A &= \int_0^{2\pi} du \int_0^{2\pi} dv \\ &= \iint_{\text{region}} du dv \sqrt{EG - F^2} \\ &= b \int_0^{2\pi} du \int_0^{2\pi} dv (a + b \cos u) \\ &= 2\pi b \int_0^{2\pi} (a + b \cos u) du \\ &= 2\pi b \left[au + b \sin u \right]_0^{2\pi} = 4\pi^2 ab \\ &= (2\pi a)(2\pi b) \end{aligned} \quad \textcircled{5}$$

Solutions to Coursework #7

Geometry II

2008

Q3

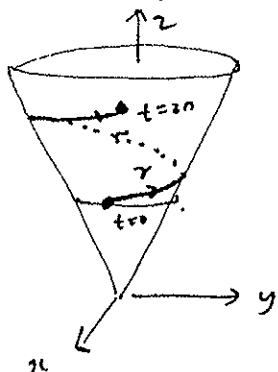
(a) using σ_u, σ_v from Q1 (wks):

$$(3) E = \|\sigma_u\|^2 = 1 + \frac{u^2}{u^2+v^2} = \frac{v^2+2u^2}{u^2+v^2}$$

$$(3) G = \|\sigma_v\|^2 = \frac{u^2+2v^2}{u^2+v^2}$$

$$(3) F = \sigma_u \cdot \sigma_v = \frac{uv}{u^2+v^2}$$

$$(b) \gamma(t) = (e^t \frac{\cos t}{c}, e^t \frac{\sin t}{s}, e^t) \quad t \in (0, 2\pi)$$



(this is a log-spiral in the $x-y$ plane, also increasing in z)

$$u = e^t \cos t, \quad v = e^t \sin t$$

the curve in the (u,v) -plane corresponds to

$$\dot{u} = e^t(c-s) \quad \dot{v} = e^t(c+s)$$

$$S(\gamma) = \int_0^{2\pi} \sqrt{E \dot{u}^2 + 2F \dot{u} \dot{v} + G \dot{v}^2} dt$$

$$= \int_0^{2\pi} \sqrt{(1+c^2)e^{2t}(c-s)^2 + (1+s^2)e^{2t}(c+s)^2 + 2cs e^{2t}(c-s)(c+s)} dt$$

or

$$(3) E = 1 + \frac{e^{2t} \cos^2 t}{e^{2t}} = 1 + \cos^2 t = 1 + c^2$$

$$(3) G = 1 + \frac{e^{2t} \sin^2 t}{e^{2t}} = 1 + \sin^2 t = 1 + s^2$$

$$(3) F = \frac{e^{2t} \cos t \sin t}{e^{2t}} = cs$$

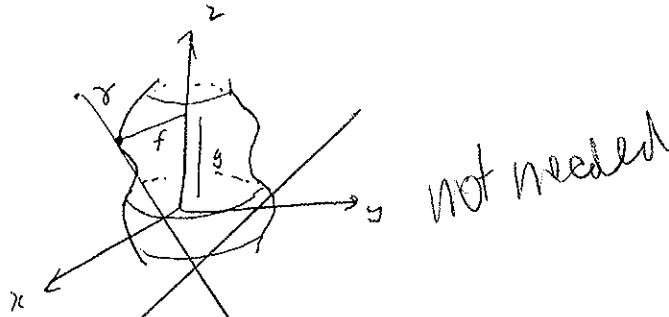
$$S(\gamma) = \int_0^{2\pi} \sqrt{(1+c^2)(c^2+s^2-2cs) + (1+s^2)(c^2+s^2+2cs) + 2cs(c^2-s^2)} dt$$

$$= \int_0^{2\pi} dt e^t \sqrt{2 + c^2 + s^2 - 2c^3s + 2s^3c + 2c^3s - 2cs^3} \quad (2)$$

$$= \int_0^{2\pi} dt e^t \sqrt{3} = \sqrt{3} [e^t]_0^{2\pi} = \sqrt{3} (e^{2\pi} - 1) \quad (2)$$

Q3

(a)



γ is a curve in the $x-z$ plane. $\sigma(u, v)$ is the rotation of this by an angle v .

$$\sigma(u, v) = (f(u) \cos v, f(u) \sin v, g(u))$$

$$\sigma_u = (f' \cos v, f' \sin v, g')$$

$$\sigma_v = (-f \sin v, f \cos v, 0)$$

$$f' \equiv \frac{d}{du} f(u)$$

$$\sigma_u \times \sigma_v = (f g' \cos v, -f g' \sin v, f f') \quad \left[\begin{array}{l} g' = \frac{d}{du} g(u) \end{array} \right]$$

$$\|\sigma_u \times \sigma_v\|^2 = f^2 g'^2 + f^2 f'^2$$

$$= f^2 (f'^2 + g'^2)$$

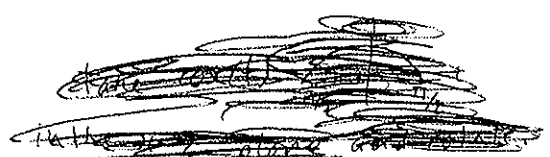
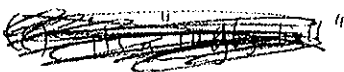
$$= f^2 \|\dot{\gamma}(u)\|^2 \quad \text{or } \dot{\gamma}(t) = \frac{d}{dt} (f, 0, g)$$

3

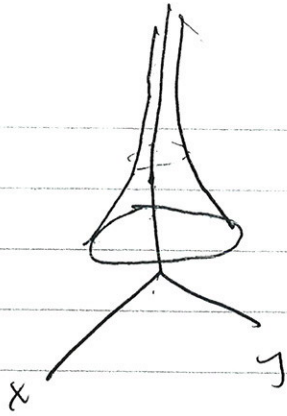
(3) regular iff $\dot{\gamma} \neq 0$ i.e. γ regular (as $f > 0$)

$$(b) \quad (3) \quad E = f'^2 + g'^2 = \|\dot{\gamma}\|^2, \quad F = \sigma_u \cdot \sigma_v = 0 \quad (3)$$

$$(3) \quad G = f^2$$



3.c)



(4)

4 a) $\tilde{\sigma}_r = (\cos \theta, \sin \theta, 2r)$

$\tilde{\sigma}_\theta = (-r \sin \theta, r \cos \theta, 0)$

$E = \tilde{\sigma}_r \cdot \tilde{\sigma}_r = 1 + 4r^2$ (3)

$F = \tilde{\sigma}_r \cdot \tilde{\sigma}_\theta = 0$ (3)

$G = \tilde{\sigma}_\theta \cdot \tilde{\sigma}_\theta = r^2$ (3)

b) $z \leq 2$ corresponds to $r \leq \sqrt{2}$ (5)

$\int_0^{\sqrt{2}} \int_0^{2\pi} \sqrt{EG - F^2} d\theta dr = \int_0^{\sqrt{2}} \int_0^{2\pi} \sqrt{(1+4r^2)r^2} d\theta dr$ (3)

$= 2\pi \int_0^{\sqrt{2}} r \sqrt{1+4r^2} dr = \frac{2\pi}{12} (1+4r^2)^{3/2} \Big|_{r=0}^{r=\sqrt{2}}$ (3)

$= \frac{\pi}{6} [(1+4 \cdot 2)^{3/2} - (1+0)^{3/2}] = \frac{\pi}{6} [9^{3/2} - 1^{3/2}]$

$= \frac{\pi}{6} [27 - 1] = \frac{26\pi}{6} = \frac{13\pi}{3}$ (5)